PERGAMON

International Journal of Solids and Structures 37 (2000) 6755–6771

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

The overall elastic moduli of orthotropic composite and description of orthotropic damage of materials

Aihong Zhao [1], Jilin Yu*

Department of Modern Mechanics, University of Science and Technology of China, Hefei, Anhui 230027, The People's Republic of China

Received 11 August 1999

Abstract

This paper presents a model of orthotropic damage of materials that combines macroscopic mechanical properties with microstructure parameters of the material. This model proposes nine elastic constants as the damage variables to describe the quantity and orientation of the damage. Based on Eshelby's equivalent principle, a simplified approach to obtaining the overall moduli for a multiphase, anisotropic composite is developed. The overall elastic compliance tensor of an orthotropic composite reinforced with three mutually perpendicular families of ellipsoidal inclusions is then derived. As special cases, explicit expressions of the overall elastic compliance tensor of a damaged material with three mutually perpendicular families of penny voids, needle voids or cracks, respectively, are presented. The relation of stress and strain with microstructure parameters is given. Moreover, the effect of microstructure parameters on damage of materials is analyzed. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Orthotropic composite; Overall elastic moduli; Damage model; Continuum and mesostructural damage

1. Introduction

Since Kachanov (1958) proposed the concept of damage mechanics, various damage mechanics models have been developed. Main methods used in the study can be put into two categories, i.e. continuum damage mechanics and meso-structural damage mechanics.

In continuum damage mechanics, the damage variables are defined as the effective surface density of microcracks in a Representative Volume Element (RVE). This, in association with the effective stress concept and the principle of equivalence, has given rise to methods of damage measurement based on

* Corresponding author. Tel: +86-551-360-2179; fax: +86-551-363-7341.
E-mail address: jlyu@ustc.edu.cn (J. Yu)

[1] Current address: State Key Laboratory of Tribology, Department of Precision Instruments and Mechanology, Tsinghua University, Beijing 100084, The People's Republic of China.

PII: S0020-7683(99)00226-7

changes in elastic or plastic properties (see, e.g. Lemaitre, 1996). These concepts have been generalized to the three-dimensional case by means of the state potential and the potential of dissipation. Accordingly, state coupling occurs between elastic strain and damage, and kinetic coupling takes place between plastic strain and damage, which allow one to calculate strains and damage up to failure if the constitutive equations for the damage are known. For example, Schapery (1990) used strain energy-like potentials to model the mechanical behavior of linear and nonlinear elastic metrics with growing damage. Theory and experiments have shown that in particulate and fibrous composite materials, the effect of strain history is often quite limited, at least for limited deformation. Zheng and Betten (1996) proposed a general description of the theory. Based on this theory, Gao et al. (1996) gave a double-scalar formula of isotropic elastic damage. However, since the irreversible thermodynamical theory gives only a thermodynamically admissible framework of the constitutive and evolution equations, the identification of the fundamental aspects of these equations should have recourse to a series of experiments. Moreover, since the constitutive equations in the traditional continuum damage theories (Kachanov, 1958; Lemaitre, 1996; Chaboche, 1984) are based on strain (or, complementary energy or stress) hypothesis, these theories have some limitations. That is, the Poisson's ratio is constant and the elongation modulus and shear modulus obey the same rule in damage evolution of materials (Gao et al., 1996). Obviously, these restrictions describe only some special cases in the damage evolution of materials.

When a material is subjected to an arbitrary load, the damage is generally anisotropic even if the undamaged material is macroscopically isotropic. For instance, an isotropic material will produce orthotropic damage when subjected to a uniaxial load due to the preferential orientation of micro defects. The mechanical behavior of anisotropic solids requires a suitable mathematical modeling. Tensor functions constitute a rational basis for a consistent mathematical modeling of complex material behaviors. A few models of anisotropic damage of materials have been proposed, which directly extend the concept of the effective stress and the damage variables to a three-dimensional stress state and describe the damage of materials by damage tensors (see, e.g. Chaboche, 1984; Sidoroff, 1981). Certain principles, methods, and applications of tensor functions on continuum damage mechanics for creep rupture was presented by Betten (1992). He also discussed the rules for specifying irreducible sets of tensor invariants and tensor generators for material tensors of rank two and four. Hayakawa and Murakami (1997) used Gibbs thermodynamic potential to describe the experimental damage surface. The change in elastic moduli due to damage development and the initial and subsequent damage surfaces expressed in the stress space are described well by the proposed theory. The existence of a damage potential and the corresponding normality law were verified experimentally. However, the restrictions mentioned above still remain.

In the case of meso-structural damage mechanics, attempts have been made to predict the macroscopic mechanical response of heterogeneous materials based on micromechanical models of the material damage. Micromechanical models have the distinct advantage of being able to capture structural details at the microscale and mesoscale, and to allow the formulation of the kinetic equations for damage evolution based on actual physical processes involved. These models, however, can be computationally inefficient in many practical applications, and can only be applied to limited classes of materials and damage mechanisms.

Both continuum damage mechanics and meso-structural damage mechanics have some advantages and disadvantages. A combination of the two approaches is an optimum way to find the damage mechanism of materials related to their microstructure and, at the same time, to the application in engineering practices. However, owing to a limitation of the two approaches, it has not come true. One of the main reasons is that the description of material damage, which is the start point for investigating damage of materials has not been well documented.

In this paper, the relation between the macroscopic properties and microstructure parameters of materials is investigated. We focus on the orthotropic damage in materials. Such a situation occurs frequently in engineering practices. For example, in a fiber-reinforced composite material, the development of damage depends on the initial mesoscopic structure of the material. Microcracks are mainly oriented parallel or perpendicular to the fiber directions and the initial symmetries are usually preserved. In such cases, it is sufficient to consider damage in three principal directions. In an initially macroscopically isotropic material with random distribution of microstructures, orthotropic damage occurs when it is subjected to a unidirectional load. In the case where orthotropic damage is dominant, the results can be regarded as the constitutive equation for an elastic metrics with damage, or for inelastic materials when the degradation of elastic moduli physically depends on damage. However, the evolution equation is not dealt with here.

The essential task for constructing the model is to determine the overall elastic moduli of a damaged material. Here we treat voids and cracks in the damaged material as special inclusions whose elastic moduli is zero. Based on Eshelby's equivalent principle (Eshelby, 1957), we first derive the effective moduli of a multiphase, anisotropic composite. The overall elastic compliance tensor of an orthotropic composite reinforced with three mutually perpendicular families of ellipsoidal inclusions is then derived. As special cases, explicit expressions of the overall elastic compliance tensor of a damaged material with three mutually perpendicular families of penny voids, needle voids or cracks, respectively, are presented. The relation of stress and strain with microstructure parameters is given. This information is important in the optimization of composites. Finally, the effect of microstructure parameters to damage of materials is analyzed. In comparison with the method used by Weng (1984), which is based on Mori–Tanaka's concept of "average stress" in the matrix and Eshelby's equivalent principle (Mori and Tanaka, 1973), the proposed formulae and the process of deduction are much simpler.

For brevity, symbolic notations will be used in the general theory. Bold face Greek letters denote the second rank tensors, and bold face capital English letters denote the fourth rank ones.

2. The model of orthotropic damage of materials

As long as material damage is considered, it is important to define proper damage variables. Here the damage variables are defined as the relative change of elastic moduli, i.e.

$$D_{11} = 1 - \frac{\tilde{E}_1}{E_0},\ D_{22} = 1 - \frac{\tilde{E}_2}{E_0},\ D_{33} = 1 - \frac{\tilde{E}_3}{E_0},$$

$$D_{12} = 1 - \frac{\tilde{\nu}_{12}}{\nu_0},\ D_{31} = 1 - \frac{\tilde{\nu}_{31}}{\nu_0},\ D_{23} = 1 - \frac{\tilde{\nu}_{23}}{\nu_0},$$

$$D_{44} = 1 - \frac{\tilde{G}_{23}}{G_0},\ D_{55} = 1 - \frac{\tilde{G}_{13}}{G_0},\ D_{66} = 1 - \frac{\tilde{G}_{12}}{G_0}. \tag{1}$$

where E_0, ν_0, G_0 are Young's modulus, Poisson's ratio and shear modulus of the undamaged material, respectively. $\tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{\nu}_{12}, \tilde{\nu}_{23}, \tilde{\nu}_{31}, \tilde{G}_{21}, \tilde{G}_{23}, \tilde{G}_{31}$ are the corresponding orthotropic elastic parameters of the damaged material. The advantage of such choice is that these damage variables have macro-mechanical meaning and they can be easily measured. Here the loss of the stiffness physically depends on the damage, i.e. the deterioration of the material. The damage variables for undamaged materials are zero. The Young's modulus and shear modulus always decrease with the increase of the density of defects in the damaged material. Some experiments (Murakami et al., 1998; Dong et al., 1997) have shown that the Poisson's ratio also decreases with the density of defects in a damaged material. In fact, the change of Poisson's ratio with the density of defects in damaged materials depends on the shape of the defects. The critical value of damage variables should be determined experimentally.

The orthotropic damage compliance matrix of the material is

$$[C] = \begin{bmatrix} 1/\tilde{E}_1 & -\tilde{\nu}_{21}/\tilde{E}_2 & -\tilde{\nu}_{31}/\tilde{E}_3 & 0 & 0 & 0 \\ -\tilde{\nu}_{12}/\tilde{E}_1 & 1/\tilde{E}_2 & -\tilde{\nu}_{23}/\tilde{E}_3 & 0 & 0 & 0 \\ -\tilde{\nu}_{13}/\tilde{E}_1 & -\tilde{\nu}_{23}/\tilde{E}_2 & 1/\tilde{E}_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\tilde{G}_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/\tilde{G}_{31} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/\tilde{G}_{12} \end{bmatrix} \tag{2}$$

where

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j}; \quad (i \neq j;\ i, j = 1, 2, 3) \ \ (i, j \text{ is not summed}) \tag{3}$$

The constitutive equation of the damaged material can be expressed by $\boldsymbol{C}$, or its inverse $\boldsymbol{L}$. We have

$$\sigma_{ij} = L_{ijkl}\varepsilon_{kl} \tag{4}$$

or

$$\varepsilon_{ij} = C_{ijkl}\sigma_{kl} \tag{5}$$

The nine damage variables account for the quantity and orientation of the damage. In order to connect the continuum damage mechanics with meso-structural damage mechanics, it is necessary to establish the relationship between the damage variables and the microstructure parameters. Once such relationship is available, the constitutive equation of the damaged material with mesostructural parameters is obtained without any hypothesis of effective stresses.

3. Elastic moduli of a multiphase composite

Consider a composite consisting of a matrix and embedded, arbitrarily oriented inclusions in spheroid shape with different moduli. The distribution of the inclusions is assumed homogeneous and well separated. The composite is assumed to have n phases of inclusion in addition to the matrix, where a "phase" is defined as a set of inclusions whose shape (or aspect ratio), orientation and elastic moduli are identical. Inclusions of the same material but with different orientation or aspect ratio are therefore classified as another phase. We denote the matrix as the 0th phase with an elastic moduli tensor of $\boldsymbol{L}_0$ and its volume fraction is c_0. The elastic moduli tensor of the rth phase is $\boldsymbol{L}_r$ with the volume fraction c_r, where $c_r = V_r/V$ and

$$\sum_{r=0}^{n} c_r = 1 \tag{6}$$

The traction at infinity is prescribed and corresponds to a uniform stress $\bar{\boldsymbol{\sigma}}$. Thus the average strains $\bar{\boldsymbol{\varepsilon}}$ produced in the composite is given by

$$\bar{\boldsymbol{\varepsilon}} = \boldsymbol{C}\bar{\boldsymbol{\sigma}} \quad \text{or} \quad \bar{\boldsymbol{\sigma}} = \boldsymbol{L}\bar{\boldsymbol{\varepsilon}} \tag{7}$$

where $\boldsymbol{C}$, with its inverse $\boldsymbol{L}$, is the overall compliance tensor of the composite. Both the elastic moduli tensor and the compliance tensor are positive definite, and assumed to possess a diagonal symmetry.

Note that

$$\bar{\boldsymbol{\varepsilon}} = \frac{1}{V}\int_V \boldsymbol{\varepsilon}\,\mathrm{d}V = c_0\bar{\boldsymbol{\varepsilon}}^{(0)} + \sum_{r=1}^{n} c_r\bar{\boldsymbol{\varepsilon}}^{(r)} \tag{8}$$

where $\bar{\boldsymbol{\varepsilon}}^{(0)}$ is the volume average strain of the matrix and $\bar{\boldsymbol{\varepsilon}}^{(r)}$ is that of the rth inclusion. Similarly, we have

$$\bar{\boldsymbol{\sigma}} = c_0\bar{\boldsymbol{\sigma}}^{(0)} + \sum_{i=1}^{n} c_r\bar{\boldsymbol{\sigma}}^{(r)} \tag{9}$$

where $\bar{\boldsymbol{\sigma}}^{(0)}$ is the volume average stress of the matrix and $\bar{\boldsymbol{\sigma}}^{(r)}$ is that of the rth inclusion and

$$\bar{\boldsymbol{\sigma}}^{(0)} = \boldsymbol{L}_0\bar{\boldsymbol{\varepsilon}}^{(0)} \tag{10}$$

Based on Eshelby's equivalent principle (Eshelby, 1957), $\bar{\boldsymbol{\sigma}}^{(r)}$ may also be expressed in terms of $\boldsymbol{L}_r$ or $\boldsymbol{L}_0$. That is,

$$\bar{\boldsymbol{\sigma}}^{(r)} = \boldsymbol{L}_r\bar{\boldsymbol{\varepsilon}}^{(r)} = \boldsymbol{L}_r\left(\bar{\boldsymbol{\varepsilon}}^{(0)} + \boldsymbol{\varepsilon}_r\right) = \boldsymbol{L}_0\left(\bar{\boldsymbol{\varepsilon}}^{(0)} + \boldsymbol{\varepsilon}_r - \boldsymbol{\varepsilon}_r^*\right) \tag{11}$$

where $\boldsymbol{\varepsilon}_r$ is the perturbed strain with respect to the volume average strain of the matrix, $\boldsymbol{\varepsilon}_r^*$ is Eshelby's equivalent transformation strain or eignstrain. In terms of Eshelby's tensors, we have

$$\boldsymbol{\varepsilon}_r = \boldsymbol{S}_r\boldsymbol{\varepsilon}_r^* \tag{12}$$

where $\boldsymbol{S}_r$ is Eshelby's tensor of the rth inclusion. The fourth rank $\boldsymbol{S}$ tensor possesses the symmetry that

$$S_{ijkl} = S_{jikl} = S_{ijlk} \tag{13}$$

When the matrix is isotropic, Eshelby's tensor is related to the shape and ratio of the inclusions if the matrix completely contacts with the inclusions. The components of $\boldsymbol{S}$ for a spheroid under the local coordinates coinciding with its principal axes are given by Eshelby (1957).

Substituting Eq. (12) into Eq. (11), we get

$$\boldsymbol{\varepsilon}_r^* = [(\boldsymbol{L}_r - \boldsymbol{L}_0)\boldsymbol{S}_r]^{-1}(\boldsymbol{L}_0 - \boldsymbol{L}_r)\bar{\boldsymbol{\varepsilon}}^{(0)} \tag{14}$$

Thus we have

$$\bar{\boldsymbol{\varepsilon}}^{(r)} = \bar{\boldsymbol{\varepsilon}}^{(0)} + \boldsymbol{\varepsilon}_r = \bar{\boldsymbol{\varepsilon}}^{(0)} + \boldsymbol{S}_r\boldsymbol{\varepsilon}_r^* = \boldsymbol{A}_r\bar{\boldsymbol{\varepsilon}}^{(0)} \tag{15}$$

where

$$\boldsymbol{A}_r = \boldsymbol{I} + \boldsymbol{S}_r[(\boldsymbol{L}_r - \boldsymbol{L}_0)\boldsymbol{S}_r + \boldsymbol{L}_0]^{-1}(\boldsymbol{L}_0 - \boldsymbol{L}_r) \tag{16}$$

and $\boldsymbol{I}$ is the fourth rank identity tensor.

Substituting Eq. (15) into Eqs. (8) and (9), we get

$$\bar{\boldsymbol{\varepsilon}} = \sum_{r=0}^{n} c_r\boldsymbol{A}_r\bar{\boldsymbol{\varepsilon}}^{(0)} \tag{17}$$

$$\bar{\boldsymbol{\sigma}} = \sum_{r=0}^{n} c_r \boldsymbol{L}_r \boldsymbol{A}_r \bar{\boldsymbol{\varepsilon}}^{(0)} \tag{18}$$

where

$$\boldsymbol{A}_0 = \boldsymbol{I} \tag{19}$$

Therefore, the overall elastic moduli tensor for the composite, according to Eq. (7), is

$$\boldsymbol{L} = \left[\sum_{r=0}^{n} c_r \boldsymbol{L}_r \boldsymbol{A}_r\right]\left[\sum_{r=0}^{n} c_r \boldsymbol{A}_r\right]^{-1} \tag{20}$$

While the overall compliance tensor is

$$\boldsymbol{C} = \left[\sum_{r=0}^{n} c_r \boldsymbol{L}_r \boldsymbol{A}_r\right]^{-1}\left[\sum_{r=0}^{n} c_r \boldsymbol{A}_r\right] \tag{21}$$

Specially, when the elastic moduli in Eq. (21) are set to zero ($\boldsymbol{L}_r = 0, r = 1, \ldots, n$), we obtain the elastic moduli for a damaged material containing ellipsoidal voids, in which

$$\boldsymbol{A}_r = \boldsymbol{I} + \boldsymbol{S}_r(\boldsymbol{I} - \boldsymbol{S}_r)^{-1} \tag{22}$$

In comparison with Weng's formula (Weng, 1984), the proposed formulae and the process of deduction are simpler. Firstly, we did not introduce a homogeneous material for comparison. Therefore, the average perturbed stress and strain in the matrix due to the presence of all inclusions need not be analyzed. Secondly, there is no need to solve the longitudinal eignstrain $\boldsymbol{\varepsilon}_r^*$. This is usually a difficult task. For instance, when the number of inclusion phases in the composite is three, in order to obtain components of $\boldsymbol{\varepsilon}_r^*$, namely, $\varepsilon_{11}^{(r)^*}, \varepsilon_{22}^{(r)^*}, \varepsilon_{33}^{(r)^*}$, $r = 1, 2, 3$, a set of nine equations must be solved. The more the number of phases the composite has, the more the number of equations must be solved.

4. Overall elastic moduli for composites containing three mutual orthotropic inclusions

4.1. Calculation model

The calculation model is shown in Fig. 1. Consider a composite containing three mutual perpendicularly aligned inclusions that are spheroid in shape, with the volume fraction of c_1, c_2, c_3, respectively. Assume the half axes of the spheroid of the rth set of inclusions, $a_i^{(r)}$ ($i = 1, 2, 3$), are along the direction of axes 1, 2 and 3, respectively. For the first set of inclusions with a volume fraction of c_1, we have $a_1^{(1)} \neq a_2^{(1)} = a_3^{(1)}$ and the aspect ratio of the inclusions is defined as $\alpha_1 = a_1^{(1)}/a_3^{(1)}$. Similar definitions hold for the other two phases. Furthermore, when the matrix and inclusions of the composite are isotropic, we have

$$L_{ijkl}^0 = \lambda_0 \delta_{ij}\delta_{kl} + \mu_0(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

$$L_{ijkl}^r = \lambda_r \delta_{ij}\delta_{kl} + \mu_r(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{23}$$

where λ_0, μ_0 and λ_r, μ_r are Lame constants for the matrix and the rth set of inclusions, respectively, and δ_{ij} is the Kroneker delta.

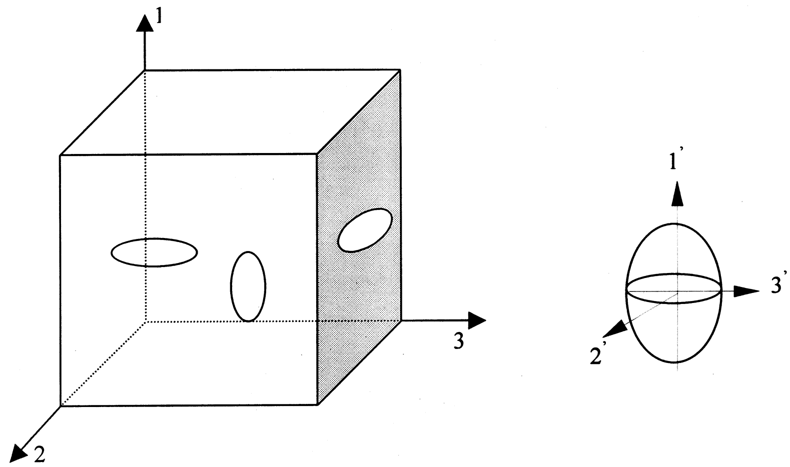

Fig. 1. Calculation model.

The software MATHEMATICA 3.0 is used to find the explicit expressions of the overall elastic moduli for the composite containing three sets of mutual orthotropic inclusions. To make use of the known results of Eshelby's $\boldsymbol{S}$ tensor under the local coordinates, conversion is made between the global coordinate system and the local coordinate system. Denote $S_{ijkl}^{(r)}$ as the Eshelby's tensor for the rth set of inclusions under the global coordinate system, we have

$$S_{ijkl}^{(1)} = S_{ijkl}$$

$$S_{1111}^{(2)} = S_{2222},\ S_{2222}^{(2)} = S_{1111},\ S_{3333}^{(2)} = S_{3333} \cdots$$

$$S_{1111}^{(3)} = S_{3333},\ S_{2222}^{(3)} = S_{2222},\ S_{3333}^{(3)} = S_{1111} \cdots \tag{24}$$

Substituting Eqs. (23) and (24) into Eq. (21), making use of Eq. (2), we have obtained elastic constants for several cases, which are given below.

4.2. Results

Expressions of the elastic moduli are summarized here according to the aspect ratio α of the inclusions. We denote the elastic constants of the rth set of inclusions as E_r, G_r and ν_r and those of the matrix as E_0, G_0 and ν_0.

4.2.1. Composite reinforced with monotonically aligned special inclusions ($c_1 \neq 0$, $c_2 = c_3 = 0$)

With the inclusions aligned along the 1-direction, there are five independent elastic constants associated with the transversely isotropic composite. They are the longitudinal Young's modulus $\tilde{E}_1$, the transverse Young's modulus $\tilde{E}_2$, the in-plane shear modulus $\tilde{G}_{12}$, the Poisson's ratio $\tilde{\nu}_{12}$, the out-of-plane shear modulus $\tilde{G}_{23}$.

4.2.1.1. Composite reinforced with thin discs ($\alpha \to 0$, $c_1 \neq 0$, $c_2 = c_3 = 0$).

$$\tilde{E}_1 = E_0 E_1 \left[c_1 E_1 (1-\nu_0) + c_0 E_0 (1-\nu_1)\right] /$$

$$\Big\{ E_0 E_1 (1-\nu_1) + c_1^2 \left[E_0(1+\nu_1) - E_1(1+\nu_0)\right]\left[E_1(1-2\nu_0) - E_0(1-2\nu_1)\right]$$

$$+ c_1 \Big[E_1^2 \left(1-\nu_0-2\nu_0^2\right) - 2E_0E_1(1-\nu_1-2\nu_0\nu_1) + E_0^2\left(1-\nu_1-2\nu_1^2\right)\Big]\Big\}$$

$$\tilde{E}_2 = \frac{[c_1E_1(1+\nu_0)+c_0E_0(1+\nu_1)][c_1E_1(1-\nu_0)+E_0(c_0+c_1\nu_1-\nu_1)]}{c_1E_1(1-\nu_0^2)+c_0E_0(1-\nu_1^2)}$$

$$\tilde{G}_{23} = \frac{1}{2}\left(\frac{c_0E_0}{1+\nu_0} + \frac{c_1E_1}{1+\nu_1}\right)$$

$$\tilde{G}_{13} = \frac{E_0E_1}{2\{E_1 + E_1\nu_0 + c_1[E_0(1+\nu_1) - E_1(1+\nu_0)]\}}$$

$$\tilde{\nu}_{23} = \frac{c_1E_1\nu_1(1-\nu_0^2) + c_0E_0\nu_0(1-\nu_1^2)}{c_1E_1(1-\nu_0^2)+c_0E_0(1-\nu_1^2)} \qquad (25)$$

4.2.1.2. Composite reinforced with circular cylinders ($\alpha \to \infty$, $c_1 \neq 0$, $c_2 = c_3 = 0$)

$$\tilde{E}_1 = \frac{c_1E_1^2[1+\nu_0+c_1(1-\nu_0-2\nu_0^2)] + c_0^2E_0^2(1-\nu_1-2\nu_1^2) + c_0E_0E_1[1+\nu_0+c_1(2-\nu_0-\nu_1-4\nu_0\nu_1)]}{E_1[1+\nu_0+c_1(1-\nu_0-2\nu_0^2)] + c_0E_0(1-\nu_1-2\nu_1^2)}$$

$$\tilde{E}_2 = \{E_0[E_1(3+c_1-4\nu_0)(1+\nu_0) + c_0E_0(1+\nu_1)][c_1E_1^2(1+\nu_0)(1+c_1(1-2\nu_0)) + c_0^2E_0^2(1-\nu_1-2\nu_1^2)$$

$$- c_0E_0E_1(1+\nu_0+c_1(2-\nu_0-\nu_1-4\nu_0\nu_1))]\}\big/\{c_0c_1E_1^3(1-\nu_0)(1+\nu_0)^3[3+2c_1-4(1+c_1)\nu_0]$$

$$+ c_0^2E_0^3[1+2c_1(1-\nu_0^2)](1+\nu_1)^2(1-2\nu_1) + c_0E_0^2E_1(1+\nu_0)(1+\nu_1)[4-6\nu_1-4\nu_0(1-2\nu_1)$$

$$+ 6c_1^2(1-\nu_0)(1-\nu_1-2\nu_0\nu_1) + c_1(1+\nu_1-\nu_0(1-\nu_1+4\nu_0\nu_1))] + E_0E_1^2(1+\nu_0)^2[3-4\nu_0+6c_1^3(1$$

$$-\nu_0)(1-\nu_0-2\nu_0\nu_1) - 2c_1(\nu_1+\nu_0(2-3\nu_1-\nu_0(3-2\nu_1))) - c_1^2(1-2\nu_1-2\nu_0(2+3\nu_1-2\nu_0(1$$

$$+2\nu_1)))]\}$$

$$\tilde{G}_{23} = \frac{E_0[E_1(3 + c_1 - 4\nu_0)(1 + \nu_0) + c_0 E_0(1 + \nu_1)]}{2(1 + \nu_0)\{c_0 E_1(3 - \nu_0 - 4\nu_0^2) + E_0(1 + \nu_1)[1 + c_1(3 - 4\nu_0)]\}}$$

$$\tilde{G}_{13} = \frac{E_0[(1 + c_1)E_1(1 + \nu_0) + c_0 E_0(1 + \nu_1)]}{2(1 + \nu_0)[c_0 E_1(1 + \nu_0) + (1 + c_1)E_0(1 + \nu_1)]}$$

$$\tilde{\nu}_{13} = \frac{E_1(1 + \nu_0)[c_0\nu_0 + 2c_1\nu_1(1 - \nu_0)] + c_0 E_0 \nu_0(1 - \nu_1 - 2\nu_1^2)}{E_1(1 + \nu_0)[1 + c_1(1 - 2\nu_0)] + c_0 E_0(1 + \nu_1)(1 - 2\nu_1)} \tag{26}$$

4.2.2. Composite reinforced with perpendicularly aligned thin discs ($\alpha \to 0$, $c_1 \neq 0$, $c_2 \neq 0$, $c_3 = 0$).

With the inclusions c_1 and c_2 aligned along the 1-direction and the 2-direction, respectively, there are nine independent elastic constants associated with the orthotropic composite. However, due to the symmetry of axes 1 and 2 in geometry, $\tilde{E}_2$, $\tilde{\nu}_{23}$, $\tilde{G}_{23}$ can be obtained easily by exchanging elastic constants and volume fraction of inclusions in the direction of axes 1 and 2 in the expressions for $\tilde{E}_1$, $\tilde{\nu}_{13}$, $\tilde{G}_{13}$, respectively. Here only the Young's moduli $\tilde{E}_1$, $\tilde{E}_3$, the shear moduli $\tilde{G}_{12}$, $\tilde{G}_{13}$ and the Poisson's ratios $\tilde{\nu}_{12}$, $\tilde{\nu}_{13}$ are given.

$$\begin{aligned}
\tilde{E}_1 = {} & E_1\{c_1c_2E_1E_2(c_1E_1 + c_2E_2)(1 + \nu_0)^2(1 - 2\nu_0) + c_0E_0^3(1 + \nu_1)(1 + \nu_2)[(1 - c_2)(1 - \nu_1)(1 - \nu_2) \\
& - c_1[(1 - \nu_1)(1 - \nu_2) - c_2(1 - \nu_1 - \nu_2)]] + E_0^2\{2c_2(1 - c_2)E_2(1 - \nu_1^2)(1 - \nu_0\nu_2) + c_1^2[c_2E_2(1 + \nu_1) \\
& \times (1 - \nu_1(1 + 2\nu_0 - \nu_2) - \nu_0\nu_2) - E_1(1 + \nu_2)[2(1 - \nu_0\nu_1)(1 - \nu_2) - c_2(2 - 2\nu_0\nu_1 - \nu_2(2 + \nu_0 \\
& + \nu_1))]] + c_1[(1 - c_2)E_1(1 + \nu_2)[2(1 - \nu_0\nu_1)(1 - \nu_2) - c_2(1 - \nu_0\nu_1 - \nu_2(1 + 2\nu_0 - \nu_1))] - c_2E_2(1 \\
& + \nu_1)[3 - 3\nu_0\nu_2 - \nu_1(3 + 2\nu_0(1 - \nu_2) - \nu_2) - c_2(2 - 2\nu_0\nu_2 - \nu_1(2 + \nu_0 + \nu_2))]]\} + E_0(1 + \nu_0) \\
& \times \{c_2^2E_2^2(1 - \nu_0)(1 - \nu_1^2) + c_1^2E_1[E_1(1 + \nu_2)((1 - \nu_0)(1 - \nu_2) - c_2(1 - \nu_0 - \nu_2)) - c_2E_2(2 + \nu_1\nu_2 \\
& - \nu_0(2 + 4\nu_1 + \nu_2))] - c_1c_2E_2[c_2E_2(1 + \nu_1)(1 - \nu_0 - \nu_1) - E_1[3 - \nu_0(3 + 2\nu_1 + 2\nu_2(1 - \nu_1)) - c_2(2 \\
& + \nu_1\nu_2 - \nu_0(2 + \nu_1 + 4\nu_2))]]\}\}\Big/\{c_1^3[E_0 - E_1(1 - 2\nu_0) - 2E_0\nu_1][E_1(1 + \nu_0) - E_0(1 + \nu_1)]^2(1 - \nu_2^2) \\
& - E_0E_1(1 - \nu_1^2)[c_2E_2(1 - \nu_0^2) + (1 - c_2)E_0(1 - \nu_2^2)] + c_1^2[E_1(1 + \nu_0) - E_0(1 + \nu_1)]\{E_0^2(2 - c_2)(1 \\
& - \nu_2^2)(1 - \nu_1 - 2\nu_1^2) - E_1(1 - \nu_0 - 2\nu_0^2)[c_2E_2(1 - \nu_1\nu_2) - E_1(1 - \nu_2^2)] + E_0[c_1E_2(1 + \nu_1)(1 - 2\nu_1)(1 \\
& - \nu_0\nu_2) + E_1(1 + \nu_2)[-(3 - \nu_1 - \nu_0(2 + 6\nu_1))(1 - \nu_2) + c_2(1 - \nu_0(1 + 2\nu_1(1 - \nu_2)) - \nu_2(1 - \nu_1))]]\} \\
& + c_1\{c_2E_1^2E_2(1 + \nu_0)^2(1 - 2\nu_0) + (1 - c_2)E_0^3(1 + \nu_1)^2(1 - 2\nu_1)(1 - \nu_2^2) - E_0E_1(1 + \nu_0)[E_1(1 + \nu_2) \\
& \times [-2(1 - \nu_0(1 + \nu_1))(1 - \nu_2) + c_2(1 - \nu_0(1 + \nu_1) - \nu_2(1 - \nu_1))] + c_2E_2(1 + \nu_1)[2 - \nu_1(2 + \nu_2) \\
& - 2\nu_0(1 - \nu_1\nu_2)]] - E_0^2(1 + \nu_1)[-2c_2E_2(1 + \nu_1)(1 - 2\nu_1)(1 - \nu_0\nu_2) + E_1[(3 - \nu_1(3 + 4\nu_0))(1 - \nu_2^2) \\
& - c_2(1 + \nu_2)(2 - 2\nu_2 - \nu_1(2 + 3\nu_0 - 2\nu_2(1 + \nu_0)))]]\}\}
\end{aligned}$$

$$\begin{aligned}\tilde{E}_3 = {} & \{E_0(1-\nu_1^2)[E_0 - c_2(E_0 - E_2 + E_2\nu_0) - (1-c_2)E_0\nu_2][E_0(1+\nu_2) - c_2(E_0 - E_2 - E_2\nu_0 + E_0\nu_2)] \\ & + c_1^2(E_0 - E_1 - E_1\nu_0 + E_0\nu_1)[E_0(E_0 - E_1 + E_1\nu_0 - E_0\nu_1)(1-\nu_2^2) + c_2[(E_0 - E_2)(E_1 - E_0) + (E_0 \\ & - E_2 - 2E_2\nu_0)(E_0\nu_1 - E_1\nu_0) + E_0\nu_2(\nu_1(E_0 + E_2) - \nu_0(E_1 + E_2)) + E_0\nu_2^2(E_0 - E_1)]] + c_1\{2E_0^2(E_1 \\ & - E_0 - E_1\nu_0\nu_1 + E_0\nu_1^2)(1-\nu_2^2) + c_2^2(E_0 - E_2 - E_2\nu_0 + E_0\nu_2)[2E_1E_2\nu_0^2 + (E_0 - E_2)(E_1 - E_0 \\ & + E_0\nu_1^2) + E_0\nu_2[E_0 - E_1 + \nu_1(E_1 + E_0)] - \nu_0[E_2(E_0 - E_1) + E_0(\nu_1(E_1 + E_2) + 2\nu_2E_1)]] \\ & - c_2E_0[E_1E_2\nu_0^2[3 + 2\nu_1(1-\nu_2) + 2\nu_2] + E_0\nu_1\nu_2[E_0 + E_1 + E_2 + \nu_2(E_1 + E_0)] + 3(E_0 - E_1)(E_2 \\ & - E_0 + E_0\nu_2^2) - E_0\nu_1^2[3E_2 - 3E_0 - \nu_2(E_0 + E_2) + 2E_0\nu_2^2] - \nu_0[2E_0E_2\nu_1^2(1-\nu_2) + \nu_2(2E_0E_1 \\ & + E_2(3E_0 - 2E_1) + 2E_0E_1\nu_2) + \nu_1(3E_0E_1 + 2E_2(E_0 - E_1) + \nu_2(2E_1E_2 + E_0(E_1 + E_2) \\ & - 2E_0E_1\nu_2))]]]\}\Big/\{E_0(1-\nu_1^2)[E_0 - c_2(E_0 - E_2 + E_2\nu_0^2) - (1-c_2)E_0\nu_2^2] - c_1[E_0(E_0 - E_1 + E_1\nu_0^2 \\ & - E_0\nu_1^2)(1-\nu_2^2) + c_2[2E_1E_2\nu_0^3 + E_0^2\nu_1\nu_2(1+\nu_2) + E_0\nu_1^2(E_0 - E_2 + E_0\nu_2) + (E_0 - E_1)(E_2 - E_0 \\ & + E_0\nu_2^2) - E_0\nu_0(E_2\nu_1(1+\nu_1) + E_1\nu_2(1+\nu_2)) - \nu_0^2(E_0E_1 + E_2(E_0 - 3E_1) + E_0(E_2\nu_1 + E_1\nu_2))]]\}\end{aligned}$$

$$\tilde{G}_{12} = \frac{E_0E_1E_2}{2\{E_2[E_1(1+\nu_0) + c_1(E_0(1+\nu_1) - E_1(1+\nu_0))] - c_2E_1[E_2(1+\nu_0) - E_0(1+\nu_2)]\}}$$

$$\tilde{G}_{13} = \frac{E_1[c_2E_2(1+\nu_0) + (1-c_2)E_0(1+\nu_2)]}{2\{E_1(1+\nu_0) + c_1[E_0(1+\nu_1) - E_1(1+\nu_0)]\}(1+\nu_2)}$$

$$\tilde{\nu}_{13} = -c_{31}\tilde{E}_1$$

$$\tilde{\nu}_{12} = -c_{21}\tilde{E}_2 \tag{27}$$

where

$$
\begin{aligned}
c_{31} = \{ & -c_1c_2E_1E_2\nu_2(1-\nu_0)^2(1-2\nu_0) - E_0(1+\nu_0)[c_1E_1[c_1\nu_1+\nu_0(1-c_1-c_2-\nu_1)] \\
& + c_2\nu_2[c_1E_1\nu_0 + E_2(1+\nu_1)((1-\nu_0)(1-\nu_1^2) - c_1(1-\nu_0-\nu_1))] \\
& - c_1E_1\nu_2^2[c_1\nu_1+\nu_0(1-c_1-2c_2-\nu_1)]] - E_0^2(1+\nu_1)(1+\nu_2)[\nu_0(1-c_1-c_2)(1-c_1-\nu_1)(1-\nu_2) \\
& + c_1\nu_1(1-c_1-\nu_2+c_1\nu_2-c_2\nu_2)]\} \big/ \{c_1c_2E_1E_2(c_1E_1+c_2E_2)(1+\nu_0)^2(1-2\nu_0) \\
& + (1-c_1-c_2)E_0^3(1+\nu_1)(1+\nu_2)[(1-c_1-\nu_2)(1-c_2)(1-\nu_1) + c_1\nu_2(1-c_2-\nu_1)] \\
& + E_0(1+\nu_0)\{c_2^2E_2^2(1-\nu_0)(1-\nu_1^2) + c_1^2E_1[E_1((1-\nu_0)(1-\nu_2^2) - c_2(1+\nu_2)(1-\nu_0-\nu_2)) \\
& - c_2E_2(2+\nu_1\nu_2-\nu_0(2+4\nu_1+\nu_2))] - c_1c_2E_2[c_2E_2(1+\nu_1)(1-\nu_0-\nu_1) \\
& - E_1[3-\nu_0(3+2\nu_1+2\nu_2(1-\nu_1)) - c_2(2-2\nu_0-\nu_0\nu_1-4\nu_0\nu_2+\nu_1\nu_2)]]\} \\
& + E_0^2\{2c_2(1-c_2)E_2(1-\nu_1^2)(1-\nu_0\nu_2) + c_1^2[c_2E_2(1+\nu_1)(1-\nu_1-2\nu_0\nu_1-\nu_0\nu_2+\nu_1\nu_2) \\
& - E_1(1+\nu_2)[2(1-c_2)(1-\nu_0\nu_1) - \nu_2(2+2\nu_0\nu_1+c_2(2+\nu_0+\nu_1))]] \\
& + c_1[E_1(1-c_2)(1+\nu_2)[2(1-\nu_2)(1-\nu_0\nu_1) - c_2(1-\nu_2-\nu_0\nu_1-2\nu_0\nu_2+\nu_1\nu_2)] \\
& - c_2E_2(1+\nu_1)[3-3\nu_1-2\nu_0\nu_1-\nu_2(3\nu_0-\nu_1-2\nu_0\nu_1) - c_2(2-\nu_1(2+\nu_0)-\nu_2(2\nu_0+\nu_1))]]\}\}
\end{aligned}
$$

$$
\begin{aligned}
c_{21} = \{ & E_0E_2(1-\nu_1^2)[c_2E_2(1+\nu_0) + (1-c_2)E_0(1+\nu_2)][c_2\nu_2+\nu_0(1-c_2-\nu_2)] \\
& - c_1^2[E_1(1+\nu_0) - E_0(1+\nu_1)](1+\nu_2)\{E_0E_2(\nu_0-\nu_1)(1-\nu_2) \\
& - c_2[E_1E_2\nu_2(1-\nu_0-2\nu_0^2) + E_0^2\nu_1(1-\nu_2-2\nu_2^2) + E_0[E_2(\nu_0-\nu_1-\nu_2(1-\nu_1-2\nu_0\nu_1)) \\
& - E_1\nu_0(1-\nu_2-2\nu_2^2)]]\} + c_1\{E_0E_2[E_1\nu_0(1+\nu_0)(1-\nu_1) \\
& - E_0(1+\nu_1)(\nu_0(2-\nu_1)-\nu_1)](1-\nu_2^2) + c_2^2(1+\nu_1)[E_2(1+\nu_0) \\
& - E_0(1+\nu_2)][E_1E_2\nu_2(1-\nu_0-2\nu_0^2) + E_0^2\nu_1(1-\nu_2-2\nu_2^2) \\
& + E_0[E_2(\nu_0-\nu_1-\nu_2(1-\nu_1-2\nu_0\nu_1)) - E_1\nu_0(1-\nu_2-2\nu_2^2)]] \\
& - c_2[E_1E_2^2\nu_1\nu_2(1+\nu_0)^2(1-2\nu_0) - E_0^3\nu_1(1+\nu_1)(1+\nu_2)^2(1-2\nu_2) \\
& + E_0E_2(1+\nu_0)[E_1(1+\nu_2)(\nu_0-2\nu_0\nu_1-2\nu_2+\nu_0\nu_2+4\nu_0\nu_1\nu_2) \\
& + E_2(1+\nu_1)(\nu_0-\nu_1-\nu_0\nu_2(1-2\nu_1))] - E_0^2(1+\nu_1)(1+\nu_2)[E_2(3\nu_0-2\nu_1 \\
& - 2\nu_0\nu_1-\nu_2(2+\nu_0-3\nu_1-4\nu_0\nu_1)) - E_1\nu_0(1-\nu_2-2\nu_2^2)]]\}\} \big/
\end{aligned}
$$

$$\{E_2\{c_1c_2E_1E_2(c_1E_1+c_2E_2)(1+\nu_0)^2(1-2\nu_0)+(1-c_1-c_2)E_0^3(1+\nu_1)(1+\nu_2)[(1-c_1-\nu_2)(1-c_2)(1-\nu_1)+c_1\nu_2(1-c_2-\nu_1)]+E_0(1+\nu_0)[c_2^2E_2^2(1-\nu_0)(1-\nu_1^2)+c_1^2E_1[E_1((1-\nu_0)(1-\nu_2^2)-c_2(1+\nu_2)(1-\nu_0-\nu_2))-c_2E_2(2+\nu_1\nu_2-\nu_0(2+4\nu_1+\nu_2))]-c_1c_2E_2[c_2E_2(1+\nu_1)(1-\nu_0-\nu_1)-E_1(3-\nu_0(3+2\nu_1+2\nu_2(1-\nu_1))-c_2(2-2\nu_0-\nu_0\nu_1-4\nu_0\nu_2+\nu_1\nu_2))]]+E_0^2[2c_2(1-c_2)E_2(1-\nu_2)(1-\nu_0\nu_2)+c_1^2[c_2E_2(1+\nu_1)(1-\nu_2-2\nu_0\nu_1-\nu_0\nu_2+\nu_1\nu_2)-E_1(1+\nu_2)(2(1-c_2)(1-\nu_0\nu_1)-\nu_2(2-2\nu_0\nu_1-c_2(2+\nu_0+\nu_1)))]+c_1[(1-c_2)E_1(1+\nu_2)(2(1-\nu_0\nu_1)(1-\nu_2)-c_2(1-\nu_2-\nu_0\nu_1-2\nu_0\nu_2+\nu_1\nu_2))-c_2E_2(1+\nu_1)(3-3\nu_1-2\nu_0\nu_1-\nu_2(3\nu_0-\nu_1-2\nu_0\nu_1)-c_2(2-2\nu_1-\nu_0\nu_1-2\nu_0\nu_2-\nu_1\nu_2))]]\}\} \tag{28}$$

4.2.3. Materials containing voids or cracks

4.2.3.1. Damaged material with three mutually perpendicular families of penny voids ($\alpha \ll 1$). In the special case of penny shaped voids (the aspect ratio $0 < \alpha \ll 1$), Eshelby's $\boldsymbol{S}$ tensor has a simpler form (Eshelby, 1957). Assuming $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$ and the volume fraction of the voids is small so the high order terms are negligible, we obtain the following approximate expressions for the nine independent elastic constants,

$$\frac{\tilde{E}_r}{E_0} = \frac{4\pi\alpha c_0}{4\pi\alpha + c_r(1+\nu_0)[16-17\pi\alpha+16(\pi\alpha-1)\nu_0]}, \qquad (r = 1, 2, 3),$$

$$\frac{\tilde{G}_{ts}}{G_0} = \frac{c_0}{c_0 + \dfrac{4(c_r+c_s)(1-\nu_0)}{\pi\alpha(2-\nu_0)} + \dfrac{16c_t(1-\nu_0)}{16-7\pi\alpha+8(\pi\alpha-2)}}, \qquad (r \neq s \neq t;\ r, s, t = 1, 2, 3),$$

$$\frac{\tilde{\nu}_{rs}}{\nu_0} = \frac{2\pi\alpha[c_r+c_s-(c_r+c_s-2)\nu_0-2(c_r+c_s)\nu_0^2]}{\nu_0\{4\pi\alpha+c_r(1+\nu_0)[16-17\pi\alpha+16(\pi\alpha-1)\nu_0]\}}, \qquad (r \neq s;\ r, s = 1, 2, 3). \tag{29}$$

4.2.3.2. Damaged material with three mutually perpendicular families of needle voids ($\alpha \to \infty$). The nine independent elastic constants are shown below.

$$\frac{\tilde{E}_r}{E_0} = \frac{c_0}{1+2(c_s+c_t)(1-\nu_0^2)}, \qquad (r \neq s \neq t;\ r, s, t = 1, 2, 3),$$

$$\frac{\tilde{G}_{ts}}{G_0} = \frac{c_0}{1+c_r+c_s+c_t(3-4\nu_0)}, \qquad (r \neq s \neq t;\ r, s, t = 1, 2, 3),$$

$$\frac{\tilde{\nu}_{rs}}{\nu_0} = \frac{\nu_0+c_t(1-\nu_0-2\nu_0^2)}{\nu_0[1+2(c_s+c_t)(1-\nu_0^2)]}, \qquad (r \neq s \neq t;\ r, s, t = 1, 2, 3). \tag{30}$$

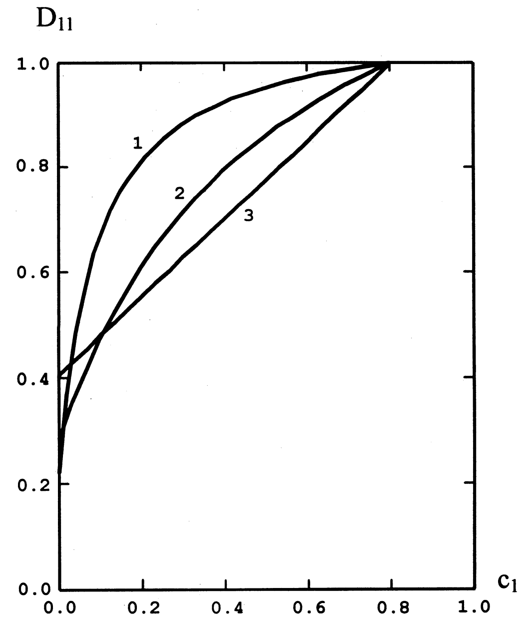

Fig. 2. Variation of D_{11} with void volume for sample 1.

4.2.3.3. Damaged materials with three mutually perpendicular families of circular cracks ($\alpha \to 0$). As the aspect ratio of the crack tends to zero, the volume fraction c_1, c_2 and c_3 are no longer adequate to describe the density of cracks. A measurable one is perhaps the crack-density parameter introduced by Budiansky and O'Connel (1976), defined as

$$\eta = \frac{Nd^3}{V}$$

where N is the number of cracks in a volume V and d is the crack diameter. Then the volume fraction for the rth set of cracks can be expressed as

$$c_r = \frac{4\pi d^2 t N_r}{3V} = 4\frac{\pi \alpha \eta_r}{3}, \qquad (r = 1, 2, 3) \tag{31}$$

where t is the thickness of the crack, $\alpha = t/d$ and η_1, η_2, η_3 are the crack densities with their normal parallel to axes 1, 2 and 3, respectively.

Setting $\alpha \to 0$ in Eq. (29), the elastic constants for the damaged material with three mutually perpendicular families of circular cracks take the forms

$$\frac{\tilde{E}_r}{E_0} = \frac{1}{1 + \frac{16}{3}(1 - \nu_0^2)\eta_r}, \qquad (r = 1, 2, 3),$$

$$\frac{\tilde{G}_{ts}}{G_0} = \frac{1}{1 + \frac{16}{3}\frac{(1 - \nu_0)}{(2 - \nu_0)}(\eta_r + \eta_s)}, \qquad (r \neq s;\ r, s = 1, 2, 3),$$

$$\frac{\tilde{\nu}_{rs}}{\nu_0} = \frac{1}{1 + \frac{16}{3}(1 - \nu_0^2)\eta_r}, \qquad (r \neq s;\ r, s = 1, 2, 3). \tag{32}$$

From Eq. (32), we get

$$\frac{\tilde{\nu}_{12}}{\tilde{E}_1} = \frac{\tilde{\nu}_{23}}{\tilde{E}_2} = \frac{\tilde{\nu}_{31}}{\tilde{E}_3} = \frac{\nu_0}{E_0} \tag{33}$$

Hence, there are three independent elastic constants for the damaged material containing three mutually perpendicular families of circular cracks, if the cracks retain their shape during the evolution. This result comes from the fact that, in the present case, there are only three independent microstructure parameters, i.e., η_1, η_2, and η_3.

5. The relationship between damage variables and microstructure of the damaged material

The explicit solutions of the damage variables can be obtained by substituting the overall elastic moduli shown above into Eq. (1), which can be expressed in general as

$$D = D(\alpha_1, \alpha_2, \alpha_3, c_1, c_2, c_3, \nu_0) \tag{34}$$

where D is one of the nine damage variables in Eq. (1). The relationship between stress and strain can then be written in the form of

$$\sigma = f(\{D\}, \varepsilon, E_0) \tag{35}$$

where $\{D\}$ denotes the assembly of all damage variables. Consequently, the dependence of the damage

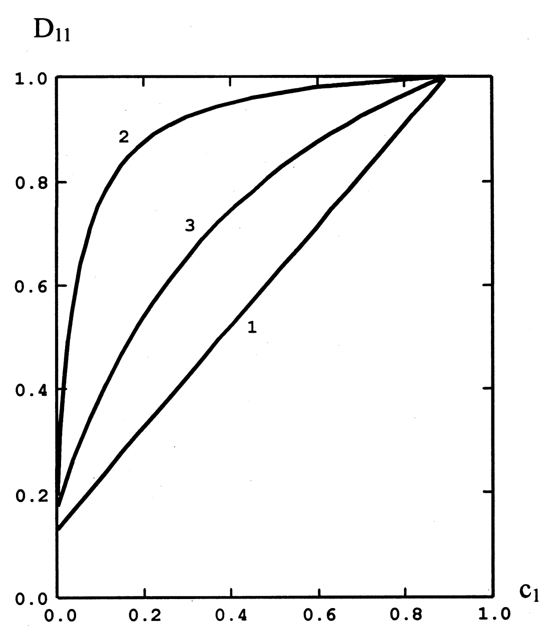

Fig. 3. Variation of D_{11} with void volume for sample 2.

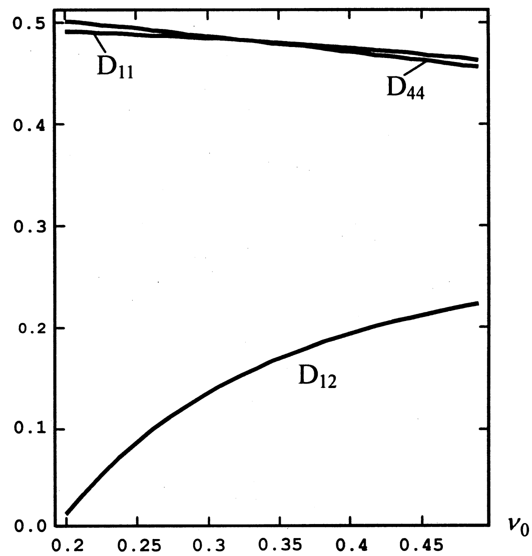

Fig. 4. Variation of damage variables with ν_0.

variables on the microstructural parameters of the material can be analyzed. Due to the limited length of the paper, we can discuss only some examples here.

5.1. Example 1

Orthotropic damages for three damaged materials having the same isotropic matrix material with $E_0 = 2.76$ GPa and $\nu_0 = 0.35$ but containing voids of different aspect ratios. The aspect ratio of voids is

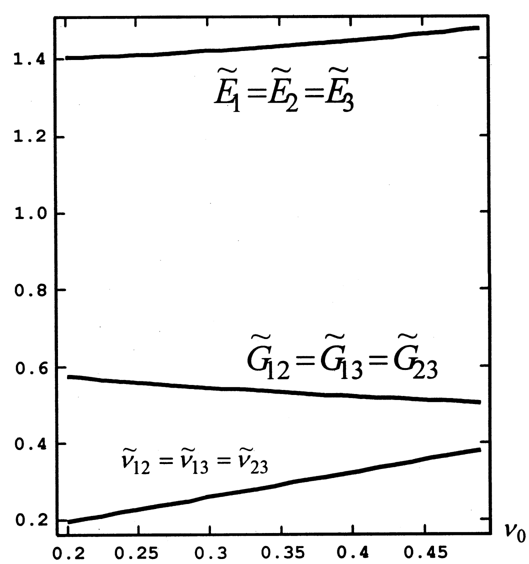

Fig. 5. Variation of overall elastic constants with ν_0.

$\alpha_1 = \alpha_2 = \alpha_3 = 0.1$, $\alpha_1 = \alpha_2 = \alpha_3 = 0.5$, and $\alpha_1 = \alpha_2 = \alpha_3 = 100$ for materials 1, 2 and 3, respectively. The variation of D_{11} with c_1 for the three materials when $c_2 = c_3 = 0.1$ is shown in Fig. 2. Comparing the three curves, the influence of the shape of voids on the damage variables is obvious.

5.2. Example 2

Orthotropic damage for three damaged materials having the same isotropic matrix material with $E_0 = 2.76$ GPa and $\nu_0 = 0.35$ but containing voids with different orientations. The aspect ratio of voids is $\alpha_1 = 25$, $\alpha_2 = 0.5$, $\alpha_3 = 0.05$ for material 1, $\alpha_1 = 0.05$, $\alpha_2 = 25$, $\alpha_3 = 0.5$, for material 2, and $\alpha_1 = 0.5$, $\alpha_2 = 0.05$, $\alpha_3 = 25$ for material 3. Fig. 3 shows the variation of D_{11} with c_1 for the three materials when $c_2 = c_3 = 0.05$. One can see the influence of orientation of voids on the damage variable.

5.3. Example 3

Orthotropic damage for materials having the same Young's modulus of the isotropic matrix material $E_0 = 2.76$ GPa, and three sets of voids with the same volume fraction, $c_1 = c_2 = c_3 = 0.1$, the same aspect ratio, $\alpha_1 = \alpha_2 = \alpha_3 = 10$. Due to the symmetry of the damaged material, we have, $D_{11} = D_{22} = D_{33}$, $D_{44} = D_{55} = D_{66}$ and $D_{12} = D_{13} = D_{23}$. To explore the effect of Poisson's ratio of the matrix apparently, the variation of each damage variable and elastic constant of damaged materials with ν_0 are shown in Figs. 4 and 5, respectively. It is observed that D_{12}, D_{13}, D_{23} are sensitive to the variation of ν_0, while $D_{11} = D_{22} = D_{33}$ and $D_{44} = D_{55} = D_{66}$ remain relatively insensitive.

6. Conclusions

Based on the proposed orthotropic damage model, the relationship between the stress tensor and strain tensor is established which requires no equivalent hypothesis. Consequently, the limitation of traditional theory is avoided. Explicit solutions of damage compliance with microstructural parameters for orthotropic damaged material containing three mutually perpendicular families of penny voids, needle voids and cracks, respectively, are presented. Accordingly, the relationship between stress and strain with microstructural parameters is established. Analysis shows that the damage variables in the proposed model always increase with the volume fraction of materials and depend on the shape and orientation of the defects. The Poisson's ratio has a greater influence on damage variables D_{12}, D_{13} and D_{23} than on the others. Nine damage variables, which are related to the changes of the elastic constants, are suggested for an orthotropic damaged material. We focused our attention on establishing the relationship between the overall elastic properties and the micro-structural parameters of the composite. By means of Eshelby's equivalent principle, a general approach to obtain the elastic moduli of a multiphase composite is proposed. More specifically, the explicit solutions of the overall elastic moduli for composites reinforced with special ellipsoidal inclusions are given. The method proposed here, however, could not be applied to non-ellipsoidal inclusions for which Eshelby's equivalent principle is not valid.

Acknowledgements

This work was supported by the major project (No. KJ951-1-201) in fundamental researches of the Chinese Academy of Sciences. The authors would also like to thank Professor Wang Zhiqiang from the Institute of Mechanics, CAS, for his help and support.

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